Number Theory - I

QF Math Circle
November 17 2025
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Question 1: (Tiling Problem): A hallway 5 meters long is to be tiled with strips of tile of widths 8 cm and 18 cm. In how many different ways can this be done without cutting some of the tiles to different widths?

Some Observations:

- This is a one-dimensional problem.
- If x strips of tile of width 8 cm and y strips of tile of width 18 cm exactly fill the length of the hallway, then $8x + 18y = 500$ and we need x, y are positive integers.

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Question 2: What is the number of pairs of positive integers (x, y) which lie on the line $8x + 18y = 500$ and in the first quadrant?

Question 3: A grocer orders apples and oranges that total $4.26. If apples cost 25 cents each and oranges cost 18 cents each, how many of each fruit did she order?

Question 4: A trucking company has to move 265 refrigerators. It has two types of trucks it can use; one carries 10 refrigerators and the other 25 refrigerators. If it only send out full trucks and all the trucks return empty, list the possible ways of moving all the refrigerators.

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Definitions: An equation for which we seek integer solutions is called a Diophantine equation.

An equation of the form $ax + by = c$ for integers x and y, when a, b, and c are given integers is called a linear Diophantine equation.

Examples

1. $8x + 18y = 500$ (Infinitely many integer solutions!)
2. $16x + 12y = 49$ (No integer solutions!)
3. $16x + 12y = 144$ (Infinitely many integer solutions!)
4. $16x - 12y = -24$ (Infinitely many integer solutions!)

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Greatest Common Divisor

Definition: Let a and b be given integers with $b \neq 0$.

If $a = b\ n$ for some integer n then we say that b is a divisor or a factor of a. We also say a is a multiple of b.

Example: $51 = 3\ \ 17$. Hence 3 is divisor of 51 or 3 divides 51. Similarly, 17 is a divisor of 51.

Definition: A positive integer greater than 1 is called a prime number if it's only positive divisors are 1 and itself.

Examples: 2, 3, 5, 7, 11,…. are the list of few primes.

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Greatest Common Divisor

Definition: When m and n are integers, the number $ma + nb$ is called an integer combination of a and b.

Question: given a and b, what numbers can be written as integer combinations of a and b?

Follow-up question: If $1 = ma + nb$, then all numbers can be written as integer combinations of a and b. For what values of a and b is this possible?

Example: $9 = 6\cdot(-15) + 3\cdot 33$, so 9 is an integer combination of -15 and 33.

Example: $1 = 3\cdot 6 + (-1)\cdot 17$, so 1 is an integer combination of 6 and 17.

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Greatest Common Divisor

Definition: Given integers a, b not both 0, the greatest common divisor gcd(a, b) is the largest natural number that divides both a and b. By convention, gcd(0, 0) = 0.

Examples: gcd(24, 36) = 12, gcd(98,126) = 14, and gcd(98, 51) = 1.

Observation: If d = gcd(a, b) ≠ 0, then a / d and b / d are relatively prime. This enable us to describe all integer combinations of a and b.

Check: 24 = 12·2 and 36 = 12·3. Check: 98 = 14·7 and 126 = 14·9

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Greatest Common Divisor

Definition: If gcd (a, b) = 1 for some nonzero integers, then a and b are called relatively prime. For example: 5 and 12.

Given two integers a and b, how can we find gcd(a, b)?

Observation: If a = q b + r for integers a, b, q, and r, then gcd(a, b) = gcd(b, r).
Proof: ?

Example:
gcd(98, 28) = gcd(28, 14) = gcd(14,0) = 14.

Example:
gcd(98,−28) = gcd(−28,14) = gcd(14,0) = 14.

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The Euclidean Algorithm

The Euclidean Algorithm

INPUT: A pair of nonnegative integers, not both 0.

OUTPUT: The greatest common divisor of the input pair.

INITIALIZATION: Set the current pair to be the input pair.

ITERATION: If one element of the current pair is 0, then report the other element as the output and stop. Otherwise, replace the maximum element of the current pair with its remainder upon division by the other element, and repeat using this new pair as the current pair.

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The Euclidean Algorithm

Example: Compute gcd(168,136).

(168,136)	$32 = 168-1\cdot 136$
(136,32)	$8 = 136-4\cdot 32$
(32,8)	$0 = 32-4\cdot 8$
(8,0)	

Example: Compute gcd(1188,420)

(1188,420)	$348 = 1188-2\cdot 420$
(420,348)	$72 = 420-1\cdot 348$
(348,72)	$60 = 348-4\cdot 72$
(72,60)	$12 = 72-1\cdot 60$
(60,12)	$0 = 60-5\cdot 12$
(12,0)	

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The Euclidean Algorithm

Important Remark: One can reverse the steps in the Euclidean Algorithm to express gcd(a, b) as an integer combination of a and b.

Example: express 8 as an integer combination of 168 and 136.

$8 = 136 - 4 \cdot 32$

$\quad = 136 - 4 \cdot (168 - 1 \cdot 136)$

$\quad = (-4) \cdot 168 + 5 \cdot 136$

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Linear Diophantine Equations

Important Remark: Since Euclidean Algorithm finds the greatest common divisor of any two integers (not both zero), we can conclude the following: If gcd (a, b) = c, then an integer solution to Linear Diophantine equation

ax + by = c can be computed by Euclidean Algorithm.

Example: Find an integer solutions of 8x +18y = 2.

gcd(18,8) = gcd (8, 2) = gcd (2,0) = 2

18 = 8 2 + 2; 8 = 4 2 + 0

Hence 2 = 18 – (8   2). x = – 2, y = 1 is an integer solution to 8x +18y = 2.

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Linear Diophantine Equations

Theorem (Linear Diophantine Equation Theorem):

i) The Linear Diophantine equation $ax + by = c$ has a solution if and only if gcd(a, b) divides c.

ii) If $\gcd(a, b) = d \neq 0$ and $x = x_0$ and $y = y_0$ is a particular solution, then the complete integer solution (set of all integer solutions) is given by

$$x = x_0 + n\,(b / d), \qquad y = y_0 - n\,(a / d), \text{ for all } n \text{ in } Z.$$

Remark: For positive integer solutions we need to solve:

$$x = x_0 + n\,(b / d) > 0 \quad \text{AND} \quad y = y_0 - n\,(a / d) > 0.$$

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Linear Diophantine Equations

Example (Back to Tiling Problem): Find the positive integer solutions of $8x + 18y = 500$.

We have seen that $x = -500$ and $y = 250$ is a particular solution to given equation. Then by the Linear Diophantine Equation Theorem the set of integer solutions to

$8x + 18y = 500$ given by $x = -500 + 9n$ and $y = 250 – 4n$, where n is in Z. For positive integer solutions we need to solve:

$$x = -500 + 9n > 0 \quad \text{AND} \quad y = 250 - 4n > 0.$$

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Linear Diophantine Equations

Example (Back to Tiling Problem):

By solving x = -500 + 9n > 0 AND y = 250 - 4n > 0,

We obtain $56 \le n \le 62$. Then all positive integer solutions are listed in the following table:

n	56	57	58	59	60	61	62
x = 9n - 500	4	13	22	31	40	49	58
y = 250 – 4n	26	22	18	14	10	6	2

Consequently there are 7 different solutions to our tiling problem.

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Linear Diophantine Equations

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Linear Diophantine Equations

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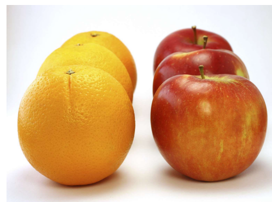

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Linear Diophantine Equations

Question 4: A trucking company has to move 265 refrigerators. It has two types of trucks it can use; one carries 10 refrigerators and the other 25 refrigerators. If it only send out full trucks and all the trucks return empty, list the possible ways of moving all the refrigerators.

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