Carnegie Mellon University Qatar

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  502884197169399375105820974944592 3
 07816406286208998628034825342117067
 9821    48086        5132
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46       09550       58223
17       25359       4081
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        21105       55964
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        9303        81964
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      284756        48233
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Pi Day Mathematics Competition

Final Round 2017

Question 1

Some water evaporates in the process of drying grapes to produce raisins. In a sample of grapes, initially the water was 80% by weight and after drying water reduces to 20% by weight. What is the ratio of the initial weight of the water to the final weight of the water in the sample?

Question 2

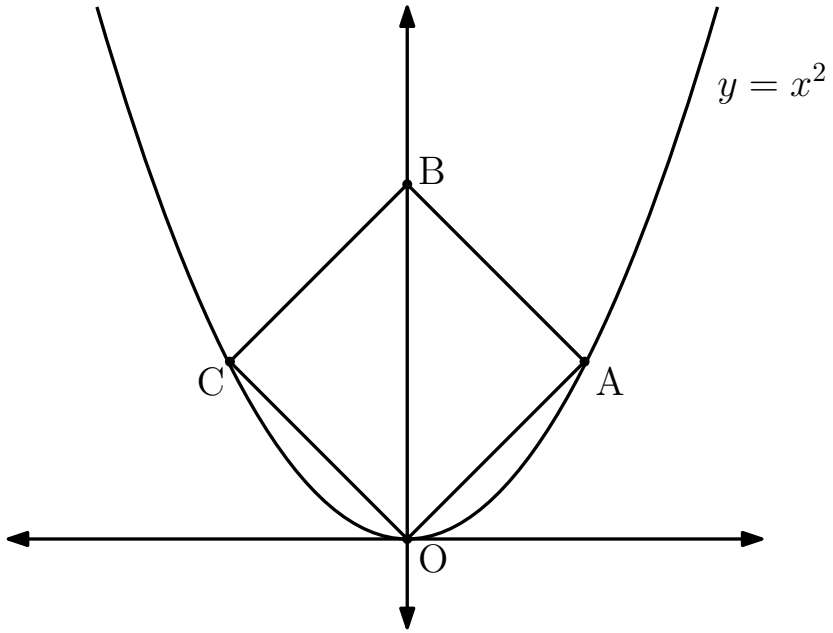
What is the remainder when 3^{2017} is divided by 5?

Question 3

In a triangle with integer side lengths, one side is five times as long as a second side, and the length of the third side is 18. What is the greatest possible perimeter of the triangle?

Question 4

In the following figure, the equation of the parabola is $y = x^2$ and $OABC$ is a square. It is given that vertex O is at the origin and vertices A and C are on the given parabola. What is the area of the square $OABC$?



Question 5

In the first round of this competition, there were 25 questions worth 1 point each, 10 questions worth 2 points each, and 5 questions worth 3 points each. Each question was a multiple choice with 5 options and a wrong answer had a penalty equal to $1/4$ of the point value of the question.

If a team randomly selects an answer for each question, what is their expected score?

Question 6

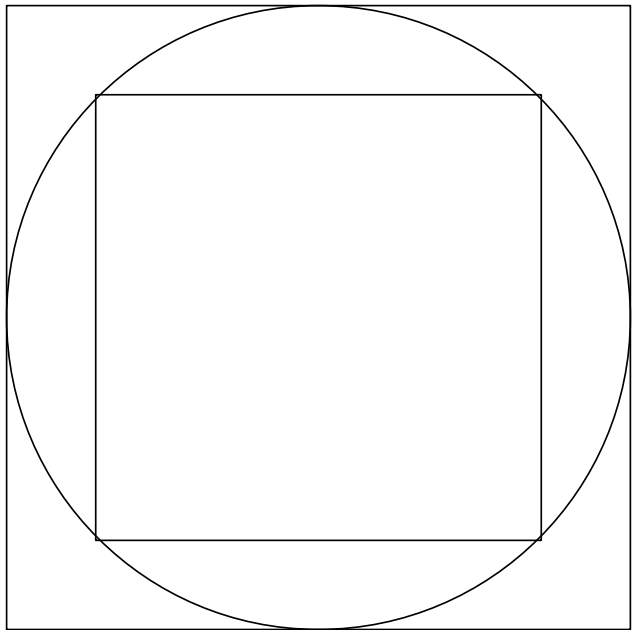
How many three digit numbers are there in which the sum of the digits is even? (We do not allow the first digit to be zero.)

Question 7

Let $f(x) = 2x + 1$ and $g(x) = \frac{2x-1}{x+5}$. If $(g^{-1} \circ f)(a) = -16$, what is the value of a?

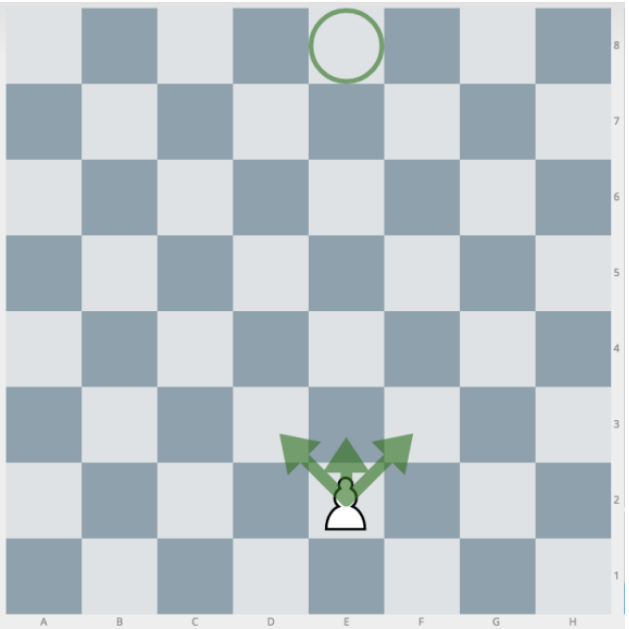
Question 8

A square is inscribed in a circe which is itself inscribed inside a square. What is the ratio of the area of the small square to that of the large square?



Question 9

On a chessboard, under the standard rules, a pawn may move to one of the three squares in front of it (the square directly in front, the square forward and to the left, or the square forward and to the right). If a pawn starts on square e2, how many different paths can it take to make it to the other side of the board (square e8)?



Question 10

Let n be a positive integer such that the remainder when it is divided by 48 is 47. If the remainder when n is divided by 49 is also 47, find the remainder when n is divided by 42.

Question 11

The number π appears in many contexts, not just in calculations in trigonometry and geometry. For example, in 1734, the mathematician Leonhard Euler showed that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

What is the value of the alternating sum

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots ?$$

Question 12

If the roots of the equation $x^2 - 5x + p = 0$ are also roots of the equation $x^3 + qx + 30 = 0$, then what is the value of the sum $p + q$?

Question 13

Find the value of $\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \frac{1}{(4)(5)} + \cdots + \frac{1}{(99)(100)}$.

Question 14

Cannibals ambush a safari in the jungle and capture three men. The cannibals give the men a single chance to escape uneaten. The captives are lined up in order of height, and are tied to stakes. The man in the rear can see the backs of his two friends, the man in the middle can see the back of the man in front, and the man in front cannot see anyone. The cannibals show the men five hats. Three of the hats are red and two of the hats are blue. Blindfolds are then placed over each man's eyes and a hat is placed on each man's head. The two hats left over are hidden. The blindfolds are then removed and it is said to the men that if one of them can guess what color hat he is wearing they can all leave unharmed. The man in the rear who can see both of his friends' hats but not his own says, "I don't know". The middle man who can see the hat of the man in front, but not his own says, "I don't know". The front man who cannot see anybody's hat says "I know!". What color was it?

Question 15

Find all positive integers $a < b < c$ such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$.

Question 16

Three friends decide to meet at a playground to play with each other. Each one of them brings two toys. At the end, they put all the toys together in a sack and then randomly take two toys each. What is the probability that the friends all end up with the same toys they brought?

Tie-breaker 1

The sequence 3, 1, 4, 1, 5, 9, 2, 6, ... is the sequence of digits of π in decimal form. What are the first five digits of π when it is expressed in binary?

Tie-breaker 2

You place 100 coins heads up in a row and number them by position, with the coin all the way on the left No. 1 and the one on the rightmost edge No. 100. Next, for every number N, from 1 to 100, you flip over every coin whose position is a multiple of N. For example, first you'll flip over all the coins, because every number is a multiple of 1. Then you'll flip over all the even-numbered coins, because they're multiples of 2. Then you'll flip coins No. 3, 6, 9, 12? And so on.

When you finish this process, which coins will be heads down?