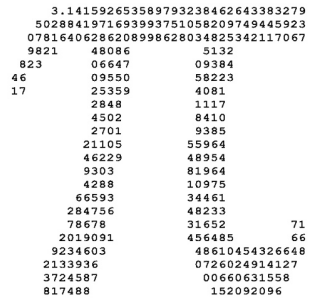
Carnegie Mellon University Qatar

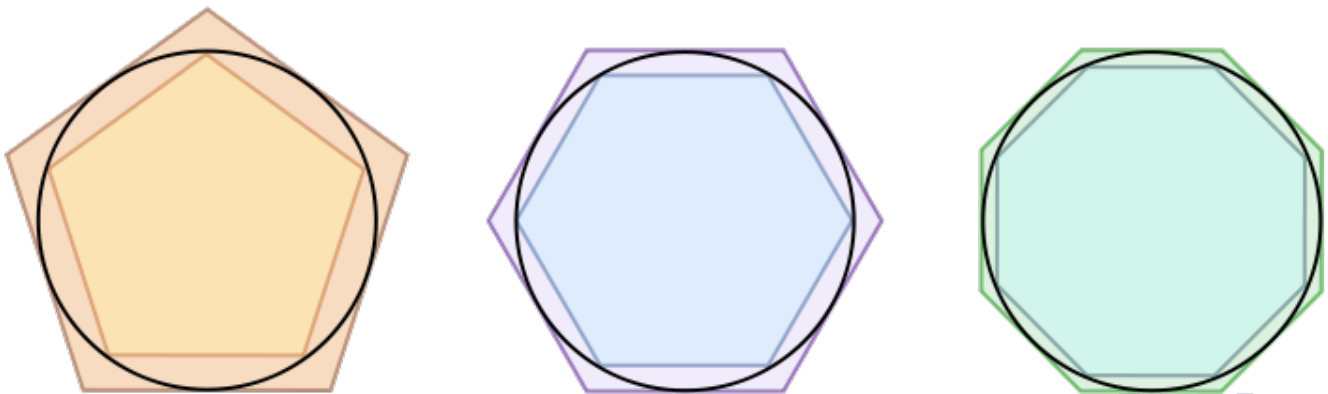


First Annual Pi Day Mathematics Competition

Final Round 2016

Approximating π

The first recorded algorithm for rigorously calculating the value of π was a geometrical approach using polygons, devised around 250 BC by the Greek mathematician Archimedes. This polygonal algorithm dominated for over 1,000 years, and as a result π is sometimes referred to as "Archimedes' constant". Archimedes computed upper and lower bounds of π by drawing a regular hexagon inside and outside a circle, and successively doubling the number of sides until he reached a 96-sided regular polygon. By calculating the perimeters of these polygons, he proved that $223/71 < \pi < 22/7$ (that is $3.1408 < \pi < 3.1429$).

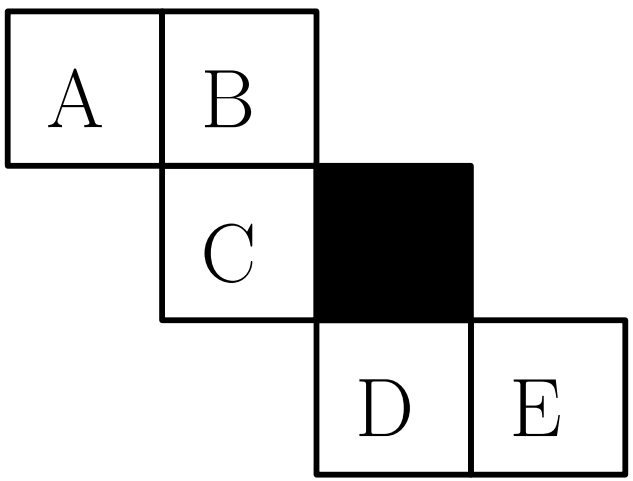


Question 1

What is the lower bound obtained when approximating using an inscribed hexagon? That is, what is the perimeter of a hexagon inscribed in a circle of diameter 1?

Question 2

Once flattened out, a perfect cube has the following shape. If the black side was facing up for the original cube, what was the letter on the bottom face of the cube?



Question 3

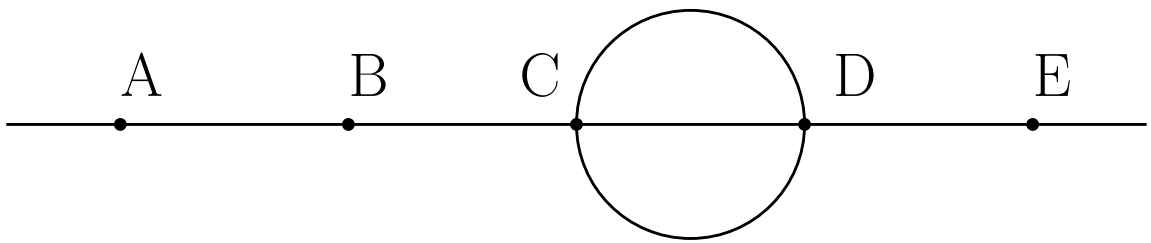
Let ABC be a triangle with $m(ABC) = 100^\circ$ and $m(CAB) = 50^\circ$. Let a, b, and c be the lengths of the sides opposite the vertices A, B, and C, respectively. What is the simplest expression (without absolute values) which is equal to $\dfrac{|a-b| + |b-c| + |c-a|}{2}$?

Question 4

The sum of 15 consecutive positive integers is 2055. What is the smallest summand? That is, what is the smallest number in the sequence?

Question 5

Let A, B, C, D, and E be points on the same line. Points C and D lie on a circle. If two points are chosen randomly among the given points, what is the probability that exactly one of the selected points will be on the circle?



Question 6

The age of a father is 26 more than the sum of the ages of his two sons. In two years, the age of the father will be twice the sum of the ages of his two sons. What is the current age of the father?

Question 7

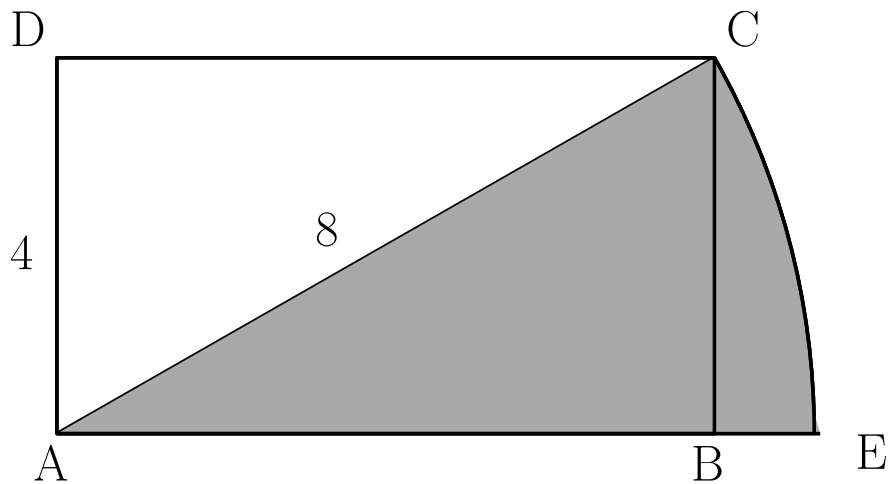
A farmer finishes a job in 30 days by doing the same amount of work every day. If he reduces the time that he works by 1/3 on every day, how long will it take him to finish the same job?

Question 8

What is the remainder when the sum $2015^{2016} + 2016^{2017} + 2017^{2018}$ is divided by 9?

Question 9

Let $ABCD$ be a rectangle and CE an arc of the circle centered at A. If $|DA| = 4cm$ and $|AC| = 8cm$,then what is the area of the shaded sector of the circle?



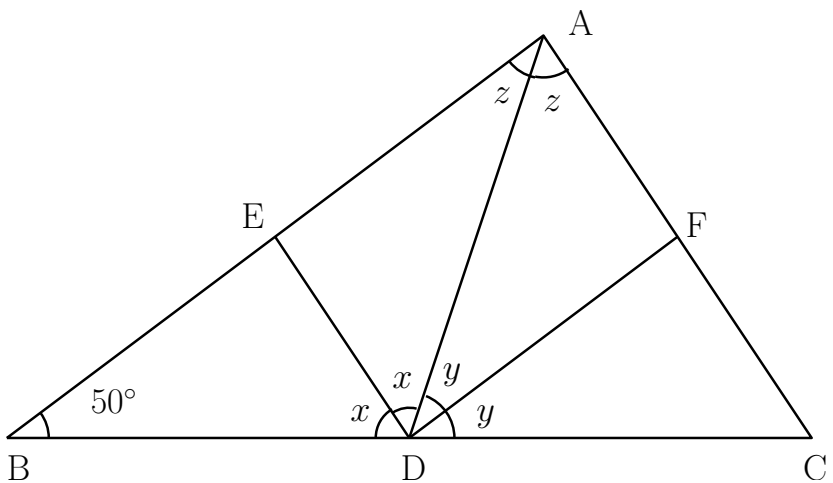
Question 10

Calculate the following sum as a decimal number:

$$\sqrt{1.44 \times 10^4} + \sqrt[3]{0.8 \times 10^{-5}} + \sqrt[4]{8.1 \times 10^5}$$

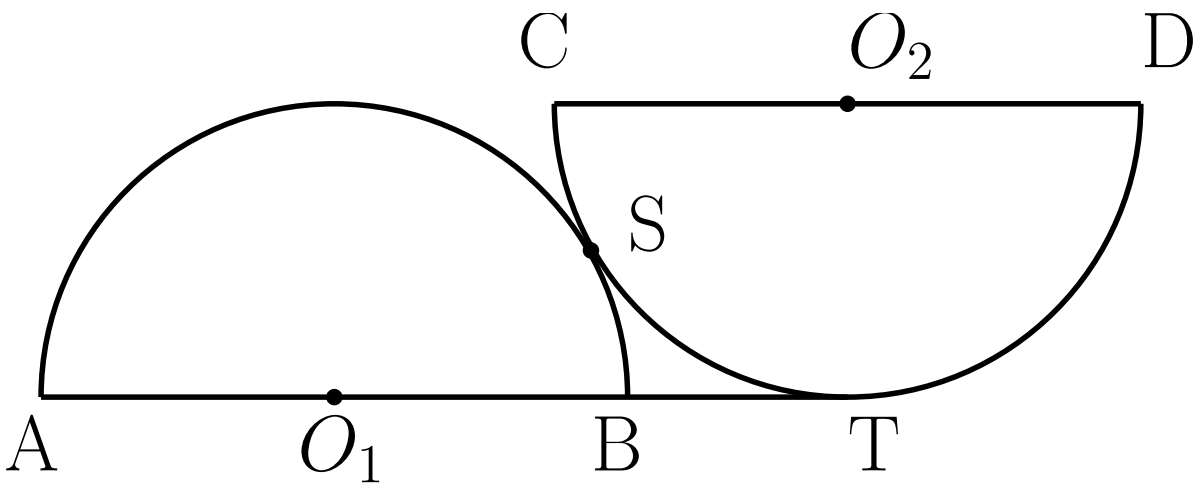
Question 11

In the figure below, $x - y = 10^\circ$. What is $m(BCA)$ in degrees?



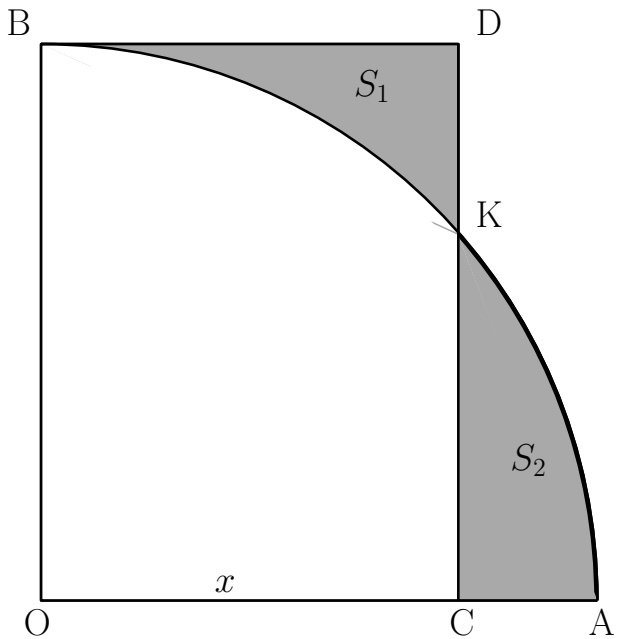
Question 12

Refer to the figure below. Assume AB is parallel to CD and O_1 and O_2 are the centers of the semicircles of radius $1cm$. The two semicircles are tangent at the point S. T is the point where the extension of line segment AB is tangent to the semicircle with center O_2. What is the length of line segment BT?



Question 13

Refer to the figure below. Let $OBDC$ be a rectangle and let BKA be an arc of a circle centered at O. Let $|OB| = |OA| = 4cm$ and $|OC| = x\,cm$. If the two shaded areas S_1 and S_2 have equal area, what is the length of x?



Question 14

Let x be an acute angle satisfying $\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = 8$. Find the value of x in degrees.

Question 15

Let a and b be positive integers. Suppose $ab = 2a + 14$. What is the value of a for which b takes its smallest possible value?

Question 16

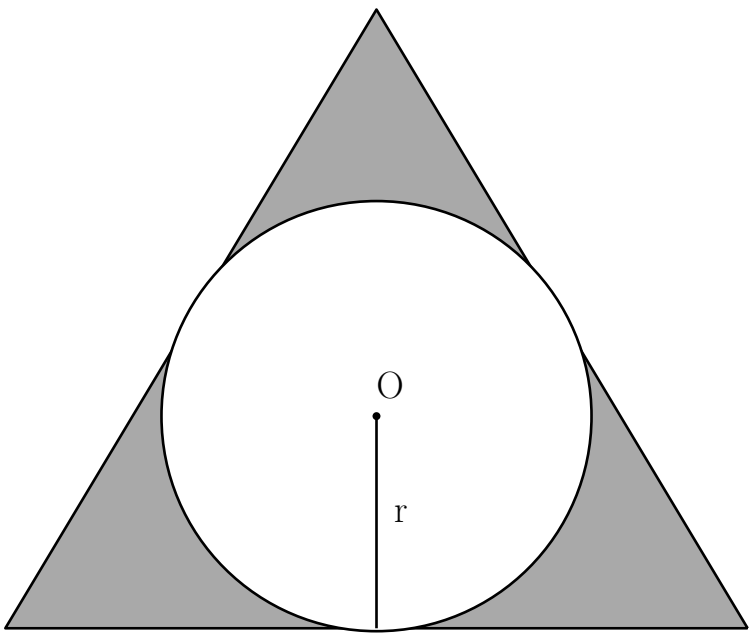
Let $x^{-a} = 2$. Find the expression $(x^{2a-1})^{-1}$ in terms of x only.

Question 17

What are the first 5 digits in the binary representation of π?

Question 18

Let ABC be an equilateral triangle with a circle inscribed in it as in the figure below. If the shaded area is $\frac{60}{3\sqrt{3}-\pi} cm^2$, what is the length of a side of the triangle?



Question 19

Find the integer which is equal to

$$\frac{2^4 \cdot 10^3}{6 + 3 \cdot 2^{-4} + 5 \cdot 2^{-4} + 3 \cdot 2^{-1}}.$$

Question 20

Let $x = 327^{95}$. What is the last/units digit of x?

Question 21

Suppose that $\frac{1}{a} + \frac{1}{b} = 1$ and $a^2 + b^2 = 24$. Find all possible products of a and b satisfying these two equations.