The story of $\pi$
and related puzzles

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Being with math is being with the truth and eternity!

Oct, 30, 2017
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Time to pause and ponder

Are we on solid ground?

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\lim_{x \to 0} \frac{\sin(x)}{x} = 1
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“Cosine rule” but Pythagoras truly rules!

The oldest, shortest words “yes” and “no” are those which require the most thought. - *Pythagoras*

---

**Cosine rule**

\[ c^2 - (b - a \cos(\theta))^2 = a^2 - (a \cos(\theta))^2 \]

\[ \Rightarrow c^2 = (b - a \cos(\theta))^2 + a^2 \sin^2(\theta) \]

\[ \Rightarrow c^2 = b^2 + a^2 - 2ab \cos(\theta) \]
Angle subtended by a side at the center of a regular \( n \)-gon \( \frac{2\pi}{n} \)

\[
\sqrt{r^2 + r^2 - 2r^2 \cos \left( \frac{2\pi}{n} \right)} = r\sqrt{2} \sqrt{1 - \cos \left( \frac{2\pi}{n} \right)} = r\sqrt{2} \sqrt{1 - \left( 1 - 2\sin^2 \left( \frac{\pi}{n} \right) \right)} = 2r\sin \left( \frac{\pi}{n} \right)
\]

\[
P_n = \frac{n \cdot 2r \cdot \sin \left( \frac{\pi}{n} \right)}{2r} = n \sin \left( \frac{\pi}{n} \right)
\]

\[
\Rightarrow \lim_{n \to \infty} \frac{P_n}{2r} = \lim_{n \to \infty} \frac{\pi \sin \left( \frac{\pi}{n} \right)}{\frac{\pi}{n}} = \pi \lim_{n \to \infty} \frac{\sin \left( \frac{\pi}{n} \right)}{\frac{\pi}{n}}
\]
• In 263 AD, Liu Hui of China using regular inscribed polygons with sides 12 to 192 showed that $3.14159 < \pi$.

• Towards the end of 5th century AD, Tsu Chung-chih and Tsu keng chih use regular polygons with 24,576 sides to show $3.1415926 < \pi < 3.1415927$. 
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Ludolph van Ceulen
Dutch-German mathematician
Ludolph van Ceulen was a German-Dutch mathematician from Hildesheim. He emigrated to the Netherlands. Wikipedia:

Born:
January 28, 1540, Hildesheim, Germany

Died:
December 31, 1610, Leiden, Netherlands

Known for: pi

Institution: Leiden University

Notable student: Willebrord Snellius
Archimedes’ approximation of $\pi$: Angle bisector

Angle bisector and ratio of sides

$$\frac{AC}{AB} = \frac{CD}{DB}$$
Archimedes’ approximation of $\pi$ (I): Upper bound

Archimedes' Approximation of $\pi$

\[
\frac{OC}{OA} = \frac{CD}{AD}
\]

\[
\frac{OC}{OA} + \frac{OA}{OA} = \frac{CD + AD}{AD} = \frac{AC}{AD}
\]

\[
\frac{OC}{AC} + \frac{OA}{AC} = \frac{OA}{AD}
\]

\[
\sqrt{\frac{O A^2 + A D^2}{A D^2}} = \sqrt{\frac{O D^2}{A D^2}} = \frac{O D}{A D}
\]
Archimedes’ approximation of $\pi$ (II)
Archimedes’ iteration

$AC' = \text{Half of the length of a circumscribing regular 6-gon}$

$$\frac{OA}{AC'} = \cot \left( \frac{\pi}{6} \right) = \frac{1}{\tan \left( \frac{\pi}{6} \right)} = \sqrt{3} > \frac{265}{153}$$

$$\frac{OC'}{AC'} = \frac{1}{\sin \left( \frac{\pi}{6} \right)} = 2 = \frac{306}{153}$$

$$\frac{OA}{AD} = \frac{OC + OA}{AC} > \frac{265}{153} + \frac{306}{153} = \frac{571}{153}$$
Archimedes’ approximation of $\pi$ (III)
Repeated use of Archimedes’ iteration

\[
\frac{OC}{AC} + \frac{OA}{AC} = \frac{OA}{AD}
\]

\[
\Rightarrow \frac{OD}{AD} + \frac{OA}{AD} = \frac{OA}{AE}
\]
Archimedes’ approximation of $\pi$ (III)
Repeated use of Achimedes’ iteration

\[
\frac{OC}{AC} + \frac{OA}{AC} = \frac{OA}{AD} \\
\Rightarrow \frac{OD}{AD} + \frac{OA}{AD} = \frac{OA}{AE} \\
\Rightarrow \frac{OE}{AE} + \frac{OA}{AE} = \frac{OA}{AF}
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\Rightarrow \frac{OF}{AF} + \frac{OA}{AF} = \frac{OA}{AG}
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\]
Archimedes’ approximation of $\pi$ (III)(b)

Pythagoras comes to rescue!

\[
\frac{571}{153} < \frac{OC}{AC} + \frac{OA}{AC} = \frac{OA}{AD}
\]

\[
\frac{591\frac{1}{8}}{153} < \sqrt{\left(\frac{(571)^2 + (153)^2}{(153)^2}\right)} < \frac{OD}{AD}
\]
Archimedes’ approximation of $\pi$ (III)(b)

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\[
\begin{align*}
\frac{571}{153} & < \frac{OC}{AC} + \frac{OA}{AC} = \frac{OA}{AD} \\
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\frac{1162}{153} & < \frac{OD}{AD} + \frac{OA}{AD} = \frac{OA}{AE}
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Archimedes’ approximation of \( \pi \) (III)(b)

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\]
Archimedes’ approximation of $\pi$: upper bound

As $\frac{OA}{AG} > \frac{4673\frac{1}{2}}{153}$, $\frac{AG}{OA} < \frac{153}{4673\frac{1}{2}}$. Thus,

$$\frac{\text{Perimeter of the circle}}{\text{diameter}} < \frac{96 \times (2 \times \text{length of AG})}{2 \times \text{length of OA}}$$
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$$< \frac{96 \times 153}{4673\frac{1}{2}}$$

$$= \frac{14688}{4673\frac{1}{2}}$$

$$= 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}}$$

$$< 3 + \frac{1}{7}$$
Archimedes’ approximation of $\pi$: Lower bound
Archimedes’ approximation of \( \pi \): Lower bound

\[
\frac{Bt}{Ct} = \frac{AB}{AC}
\]

\[
\frac{AC}{Ct} = \frac{BD}{Dt} = \frac{AD}{BD}
\]

\[
\frac{AD}{BD} = \frac{AC}{Ct} = \frac{AB + AC}{Bt + Ct}
\]
• French mathematician Viete (1540–1603) and later in 1650 John Wallis found infinite products for π.

• By 1682, James Gregory and Leibniz found a famous “useless” series:

\[ \arctan(t) = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} + \cdots. \]
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Time line III (b): series expressions for $\pi$

- Useless for 10000 terms are required to get four accurate digits! To compute 100 digits "you need to add up more terms than there are particles in the universe" [Blanter, page 42].

- In 1706, an English professor of Astronomy, John Machin using $\arctan(x) + \arctan(y) = \arctan(\frac{x+y}{1-xy})$ found:
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\[
\frac{\pi}{4} = \arctan\left(\frac{120}{119}\right) - \arctan\left(\frac{1}{239}\right) = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).
\]
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- $\frac{\pi}{4} = 4 \left(\frac{1}{5} - \frac{1}{3(5)^3} + \frac{1}{5(5)^5} - \frac{1}{7(5)^7} + \cdots\right)$
  $\quad - \left(\frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \frac{1}{7(239)^7} + \cdots\right)$. 
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$$\frac{\pi}{4} = 4 \left(\frac{1}{5} - \frac{1}{3(5)^3} + \frac{1}{5(5)^5} - \frac{1}{7(5)^7} + \cdots\right) - \left(\frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \frac{1}{7(239)^7} + \cdots\right).$$
Using the above formula Machin could calculate $\pi$ accurately till 100 places by hand!

Using the same above formula many mathematicians for next 150 years found more and more digits of $\pi$. 

In 1873, William Shanks used the formula to calculate 707 digits of which only the first 527 were correct. [Berggren, page 627]

In 1761 to 1776, Lambert and Legendre proved that $\pi$ is not a ratio of two integers. [Cajori, page 246]

In 1882, Ferdinand von Lindemann proved transcendence of $\pi$ (i.e., squaring the circle is impossible). [Berggren, page 407]
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• In 1873, William Shanks used the formula to calculate 707 digits of which only the first 527 were correct. [Berggren, page 627]
• In 1761 to 1776, Lambert and Legendre proved that $\pi$ is not a ratio of two integers. [Cajori, page 246]
• In 1882, Ferdinand von Lindemann proved transcendence of $\pi$ (i.e., squaring the circle is impossible). [Berggren, page 407]
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Viete-Wallis series (I): found infinite products for $\pi$

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$= 2^2\sin\left(\frac{x}{2^2}\right)\cos\left(\frac{x}{2^2}\right)\cos\left(\frac{x}{2^2}\right)$$
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[n-times application yields]
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[n-times application yields]

\[ \vdots \]
\[ = 2^n \sin\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right) \cdots \cos\left(\frac{x}{2^n}\right) \]
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\vdots
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\[
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\]

\[
\Rightarrow \lim_{n\to\infty} \frac{\sin(x)}{x} = \lim_{n\to\infty} \frac{\sin\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\cdots\cos\left(\frac{x}{2^n}\right)}{\frac{x}{2^n}}
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\]
As \( \lim_{t \to 0} \frac{\sin(t)}{t} = 1 \), we get

\[
\frac{\sin(x)}{x} = \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{2^2}\right) \cos\left(\frac{x}{2^3}\right) \cdots
\]

Using \( \cos(\theta) = \sqrt{\frac{1+\cos(2\theta)}{2}} \) and the above infinite product at \( x = \frac{\pi}{2} \),
Viete-Wallis series (II): found infinite products for $\pi$

As $\lim_{t \to 0} \frac{\sin(t)}{t} = 1$, we get

$$\frac{\sin(x)}{x} = \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\cos\left(\frac{x}{2^3}\right) \cdots$$

Using $\cos(\theta) = \sqrt{\frac{1 + \cos(2\theta)}{2}}$ and the above infinite product at $x = \frac{\pi}{2}$,

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2 + \sqrt{2}}}{2} \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots$$
As \( \lim_{t \to 0} \frac{\sin(t)}{t} = 1 \), we get

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\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2 + \sqrt{2}}}{2} \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots
\]
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Formal References


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Q: What will a logician choose: a half of an egg or eternal bliss?
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A: A half of an egg! Because nothing is better than eternal bliss, and a half of an egg is better than nothing.
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