This written homework covers big-\( O \) notation and some reasoning about searching and sorting algorithms. You will use some of the functions from the arrayutil.c0 library that was discussed in lecture in this assignment.

Print out this PDF double-sided, staple pages in order, and write your answers \textit{neatly} by hand in the spaces provided.

The assignment is due by 1:30pm on Monday 8\textsuperscript{th} February, 2016.

You can hand in the assignment at the beginning of class.

You must hand in your homework yourself; do not give it to someone else to hand in.
1. Another Sort

Consider the following function that sorts the integers in an array, using swap and is_sorted from arrayutil.c0.

```c
void sort(int[] A, int n)
//@ requires 0 <= n && n <= \length(A);
//@ ensures is_sorted(A, 0, n);
{
    for (int i = 0; i < n; i++)
    //@ loop_invariant 0 <= i && i <= n;
    //@ loop_invariant le_segs(A, 0, n-i, A, n-i, n);
    //@ loop_invariant is_sorted(A, ____, ____);
    {
        int s = 0;
        for (int j = 0; j < n-i-1; j++)
        //@ loop_invariant 0 <= j && j <= n-i-1;
        //@ loop_invariant ge_seg(A[j], A, 0, j);
        //@ loop_invariant s > 0 || (s == 0 && is_sorted(A, 0, j));
        {
            if (A[j] > A[j+1]) {
                swap(A, j, j+1); // function that swaps A[j] and A[j+1]
                s = s + 1;
            }
        }
        if (s == 0) return;
    }
}
```

(a) Complete the missing loop invariant on line 8.

```
//@ loop_invariant is_sorted(A, _____________, ______________);
```

(b) The asymptotic complexity of this sort depends on the number of comparisons made between pairs of array elements. Let $T(n)$ be the worst-case number of such comparisons made when sort(A, n) is called. Give a closed form expression for $T(n)$. Then express $T(n)$ in big O notation in its simplest, tightest form.

```
T(n) = ____________________________
T(n) ∈ O(__________________________)
```
(c) Using big-O notation, what is the best case asymptotic complexity of this sort as a function of $n$? Under what condition does the best case occur?

The best case asymptotic complexity of this sort is $O(\text{__________}).$ This occurs when

2. Big O notation

Recall the definition for big O:

$g \in O(f)$ if there is a constant $c > 0$ and some $n_0$ such that for every $n \geq n_0$ we have $g(n) \leq c \cdot f(n)$.

(a) Demonstrate graphically that $3n^2 + 7n + 4 \in O(n^2)$ by finding values for $c$ and $n_0$ that satisfy the definition above, where $g(n) = 3n^2 + 7n + 4$ and $f(n) = n^2$. Draw a picture to illustrate that $cn^2$ acts as an upper bound to $3n^2 + 7n + 4$ for all $n_0 \geq n$ using your values for $c$ and $n_0$. Be sure to label the horizontal and vertical axes.

$c = \text{___________________________}$

$n_0 = \text{___________________________}$
(b) In one sentence, explain why \(3n^2 + 7n + 4 \notin O(n)\).

(c) Determine the asymptotic complexity of the following function using big O notation as a function of \(a\) and \(b\) in its simplest, tightest form.

```c
int mystery(int a, int b)
//@requires a > 0 && b > 0;
{
    int x = 0;
    for (int i = 1; i < 10; i++) {
        int j = a;

        while (j > 0) {
            int k = b;

            while (k > 0) {
                x = x + i * j * k;
                k = k / 2;
            }
            j--;    
        }
    }
    return x;
}
```

\(O(\underline{\quad})\)

Reminder: as discussed in recitation, in this class when we are dealing with logarithms, we consider the "simplest form" to be the one without the base. Therefore, we prefer \(O(\log n)\) over big-\(O\) descriptions like \(O(\log_4 n)\) or \(O(\ln n)\).
3. Binary Search

Consider the search function we were analyzing from the previous written assignment. This function returned the index of the first occurrence of \( x \) in the array \( A \), or -1 if \( x \) is not found. Now let’s fill in the loop body with code that implements the binary search algorithm (on a sorted array, of course).

```c
int search(int x, int[] A, int n)
//@requires 0 <= n && n <= \\length(A);
//@requires is_sorted(A, 0, n);
/*@ensures (\result == -1 && !is_in(x, A, 0, n))
|| (0 <= \result && \result < n
&& A[\result] == x
&& (\result == 0 || A[\result-1] < x)); @*/
{
  int lo = 0;
  int hi = n;
  while (lo < hi)
  //@loopInvariant 0 <= lo;
  //@loopInvariant lo <= hi;
  //@loopInvariant hi <= n;
  //@loopInvariant gt_seg(x, A, 0, lo);
  //@loopInvariant le_seg(x, A, hi, n);
  {
    if (A[lo] == x) return lo;
    int mid = lo + (hi-lo)/2;
    if (A[mid] < x) lo = mid+1;
    else { /*@assert(A[mid] >= x); @*/
      hi = mid;
    }
  }
  //@assert lo == hi;
  if(lo != n && A[lo] == x) return lo;
  return -1;
}
```

(a) Argue that the loop has to terminate. Follow the format for termination arguments described in the first written homework!
(b) Prove that, in the case that the code returns on line 18, the postcondition on lines 4–7 always evaluates to true. We’ve given the starting facts you’ll use in this proof.

When we start an iteration of the loop, we know the following:

- \(0 \leq n \leq \text{\texttt{\text{length}}}(A)\) by the function’s precondition (line 2, \(A\) and \(n\) are never modified by the function)
- \(A[0,n)\) sorted by the function’s precondition (line 3, \(A\) and \(n\) are never modified by the function and \(A\) is not written to anywhere)
- \(\text{\texttt{\text{lo}} < \text{\texttt{hi}}}\) by the loop guard (line 11)
- \(0 \leq \text{\texttt{lo}} \leq \text{\texttt{hi}} \leq n\) by the first three loop invariants (lines 12–14)
- \(\text{\texttt{lo}} == 0 || x > A[\text{\texttt{lo}}-1]\) by the fourth loop invariant (line 15)
- \(\text{\texttt{hi}} == n || x <= A[\text{\texttt{hi}}]\) by the fifth loop invariant (line 16)
(c) Modify the search function above so that it uses the binary search algorithm to return the index of the last occurrence of $x$ in array $A$ (as opposed to the first occurrence) or -1 if $x$ is not found. Think carefully about the contracts to make sure that your array accesses are safe and that the function is logically correct!

```c
int search(int x, int[] A, int n)
//@requires 0 <= n && n <= \length(A);
//@requires is_sorted(A, 0, n);
/*@ensures (\result == -1 && !is_in(x, A, 0, n))
|| (0 <= \result && \result < n
   && A[\result] == x
   && ________________________); @*/
{
  int lo = 0;
  int hi = n;
  while (lo < hi)
    //@loop_invariant 0 <= lo;
    //@loop_invariant lo <= hi;
    //@loop_invariant hi <= n;

    //@loop_invariant ______________________;

    //@loop_invariant ______________________;
    {
      int mid = lo + (hi-lo)/2;

      if (______________________________) lo = mid+1;
      else { /*@assert(__________________________); @*/
        hi = mid;
      }
    }
    //@assert lo == hi;

    if (______________________________) {
      return ________________________________;
    }

  return -1;
}
```